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STRESS CONCENTRATION NEAR THE APEX OF A CRACK BY THE COUPLE-STRESS THEORY OF ELASTICITY AND THE METHOD OF PHOTOELASTICITY

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The stress concentration near the apex of a crack in a transverse field of simple tension has already been the object of investigations within the framework of couple-stress elasticity theory [1-5]. It follows from [1-3] that the presence of couple-stresses results in a rise in the stress concentration near the crack apex. This result "contradicts" the reducing effect of couple-stresses in the known problem about the stress concentration near a circular hole in a tensile field. It is said in [4] that the presence of couple stresses does not influence the magnitude of the stress-intensity factor, and the stress concentration near an elliptical hole in a field of simple tension is considered in [5]. There results from an analysis of this paper that the stress concentration diminishes with the increase in a new elastic constant of the material l introduced by the couple-stress theory of elasticity.

Experimental papers in which the effect of the influence of couple-stresses on the stress concentration near a crack would be clarified are still nonexistent judging by the literature.

This paper is devoted to a clarification of the effect of the influence of couple stresses on stress concentration near a crack, both analytically and experimentally by the method of photoelasticity.

§1. The stress concentration near the apex of a crack in a transverse field of simple tension is considered in a coordinate system (Fig. 1).

It is useful to introduce the sum of normal stresses (invariant) into consideration, this sum having the same value in classical and couple-stress theory and being developed from the solution of the Dirichlet problem [6, 7], in order to determine the stress-intensity factor by means of couple-stress elasticity theory. It follows from the expression for the stresses [8] that

$$\sigma_x + \sigma_y = k(2/r)^{1/2}\cos(\theta/2), \tag{1.1}$$

where k is the stress-intensity factor.

Let us introduce the complex variable $z = x + iy = z_1 + re^{\theta i}$, where z_1 is a quantity characterizing the position of the crack apex. Then (1.1) can be written in the form

$$\sigma_x + \sigma_y = \mathrm{Re}\left[k\left(\frac{2}{z - z_1}\right)^{1/2}\right]. \tag{1.2}$$

Let us apply the complex variable method. According to [6, 7], the dependence between the left side of (1.2) and the stress functions has the form

$$\sigma_x + \sigma_y = 4\mathrm{Re}[\varphi'(z)]. \tag{1.3}$$

Comparing (1.2) and (1.3) and keeping in mind that (1.2) is valid only for values of z near z_1, we obtain the expression for k in the form

$$k = 2\sqrt{2}\lim_{z\to z_1}(z - z_1)^{1/2}\varphi'(z). \tag{1.4}$$

It is seen from (1.4) that the stress-intensity factors k are determined sufficiently simply if only the function $\varphi'(z)$ is known near the crack apex.

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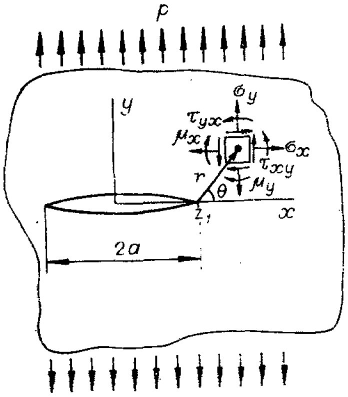

Fig. 1

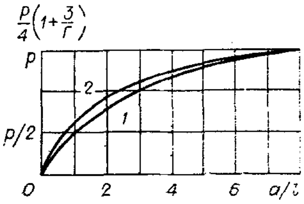

Fig. 2

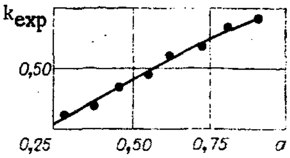

Fig. 3

It is expedient to use conformal mapping in this problem. Let $z = \omega(\zeta)$; then

$$\varphi'(z) = \frac{d\varphi}{d\zeta}\frac{d\zeta}{dz} = \frac{\varphi'(\zeta)}{\omega'(\zeta)}.$$

The point z_1 corresponding to the crack apex on the z plane hence goes over into the point ζ_1 on the ζ plane. The expression (1.4) becomes

$$k = 2\sqrt{2}\lim_{\zeta\to\zeta_1}[\omega(\zeta)-\omega(\zeta_1)]^{1/2}\frac{\varphi'(\zeta)}{\omega'(\zeta)}. \quad (1.5)$$

Let us take the mapping $z = \omega(\zeta) = (a/2)(\zeta + 1/\zeta)$ which converts a straight crack of length $2a$ along the x axis centrally located relative to the origin in the z plane into a circular hole of unit radius in the ζ plane. The point $z = a$ corresponding to the crack apex hence goes over into the point $\zeta = 1$ in the ζ plane. The expression (1.5) is converted taking into account that $\varphi'(\zeta)$ on the circular boundary has no singularity at $\zeta = 1$ and becomes

$$k = \frac{2}{\sqrt{2}}\varphi'(\zeta)\Big|_{\zeta=1}. \quad (1.6)$$

The boundary conditions of the problem at infinity are $\sigma_y^{(\infty)} = p$, $\sigma_x^{(\infty)} = \tau_{xy}^{(\infty)} = \tau_{yx}^{(\infty)} = 0$, while the couple-stresses taking account of the relation between the moments and the force stresses [7] equal

$$\mu_x^{(\infty)} = \mu_y^{(\infty)} = 0.$$

The boundary conditions on the unit circle $t = e^{i\theta}$ are

$$\sigma_r - i\tau_{r\theta} = 0,\ \mu_r = 0. \quad (1.7)$$

After the transformation of variables, the boundary conditions (1.7) acquire the form

$$\overline{\omega'(t)}\varphi(t) + \omega(t)\varphi'(t) + \overline{\omega'(t)}\overline{\psi(t)} + 8(1-\nu)l^2\overline{\varphi''(t)} - 2i\partial H/\partial\bar{t} = 0,$$
$$\mathrm{Re}\{(i/t)[8(1-\nu)l^2\overline{\varphi''(t)} - 2i\partial H/\partial\bar{t}]\} = 0,$$

where φ, ψ, H are the Kolosov–Muskhelishvili–Mindlin potentials [7]; ν is the Poisson ratio; and l is a new elastic constant introduced by couple-stress theory.

The desired stress function $\varphi(\zeta)$ is written in the form

$$\varphi(\zeta) = (pa/8)(\zeta - 3/\Gamma\zeta),$$

where $\Gamma = 1 + \frac{64(1-\nu)K_1(a/2l)}{K_1(a/2l)+K_3(a/2l)}\left(\frac{l}{a}\right)^2$; $K_1(a/2l)$ and $K_3(a/2l)$ are modified Bessel functions of the second kind (MacDonald functions).

Using the expression (1.6) to estimate stress concentration, we obtain

$$k = (p\sqrt{a}/4)(1 + 3/\Gamma). \quad (1.8)$$

Let us analyze (1.8). In the limit case [$l = 0$, $\Gamma = 1$, the point $k = (p\sqrt{a}/4)(1 + 3) = p\sqrt{a}$] we obtain the classical solution [8]. The quantity l increases with the growth of Γ and the stress-intensity factor k diminishes. In the limit case $l \to \infty$; then $k = p\sqrt{a}/4$. The change $k/\sqrt{a} = (p/4)(1 + 3/\Gamma)$ as a function of the ratio

a/l is shown in Fig. 2 for values of the Poisson ratio $\nu = 0$ and 0.3 (curves 1 and 2, respectively).

§2. As is known, the couple-stress theory of elasticity yields the most perceptible corrections to the solutions of classical elasticity theory for those problems in which the stress state has large gradients.

Among this class of problems is indeed the desired problem of the stress concentration near a crack apex. The formulation of the experiment and its idea are based on examining the stress concentration near a crack of different length in a stretched strip.

A polarization optical method at room temperature was used to determine the stress concentration. The strip specimens were fabricated from ÉD-6 epoxy resin of 1.5 ÷ 2.2 mm thickness. Cracks of different length were cut into the specimens by a sharp knife at the temperature of the highly elastic state (on the order of 100-120°C). The cracks were separated from each other and from the strip edges by a sufficiently large spacing so as to realize a uniformly stretched field far from the crack. The origination of stresses was not allowed under mechanical treatment. The loading unit (a press constructed specially for these tests) can be mounted on the KSP-7 and PPU-7 polarization optical apparatuses (fabricated by A. A. Zhdanov Leningrad State University) for the investigations.

Purely tensile forces were chosen so that the specimens would have sufficient maximal stresses while remaining in the elastic stage of operation; the limit of proportionality of the material was 550 kg/cm^2, the elastic modulus was 40,000 kg/cm^2, and the Poisson ratio was $\nu = 0.40$. Upon selecting the tensile forces, the strain was checked by a stepwise rise in the load with a 30-min hold at the end of each step. The specimen was checked as soon as the influence of creep became noticeable. The loading was cut off in the greatest strain was not more than 60% of the proportionality limit corresponding to the strain.

Near the crack apex the normal stress σ_y is expressed as

$$\sigma_y = \frac{k}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right). \tag{2.1}$$

The stress-intensity factor can be determined in terms of σ_y for $\theta = 0$ from (2.1), i.e.,

$$k = \lim_{\substack{r\to 0\\ \theta=0}} \sigma_y \sqrt{2r}. \tag{2.2}$$

The quantity σ_y in the expression (2.2) is a function of r. For cracks with a small, but nonzero apex radius, the stresses will be finite for $r = 0$ and the expression (2.1) will not be satisfied. However, near the crack apex itself the stresses can be approximated by the expression (2.1) (see [8], for example).

Near the crack apex $\sigma_{y\max}$ can be evaluated from the maximum order of the interference fringes $n_{\max}$ by using the fundamental law of photoelasticity in the form $\sigma_{y\max} = Cn_{\max}$, where C is a constant of the specimen material of given thickness.

A specimen having several cracks of different lengths was tested according to the comparative method of this investigation. The stress-intensity factors determined experimentally for any two cracks i and j were hence connected by the relationship

$$k_{i\,\exp}/k_{j\,\exp} = \Delta(n_{\max})_i/\Delta(n_{\max})_j,$$

where $\Delta(n_{\max})$ is an increment on the order of the isochrome on the contour of the crack apex corresponding to the increment in the applied load.

The high accuracy of this comparative method, which the author applied in investigating the stress concentration near a hole and an inhomogeneity [9], consists in using the ratio $\Delta(n_{\max})_i/\Delta(n_{\max})_j$, whereupon the errors due to edge effects, extrapolation, and other factors can be reduced to a minimum.

The difference in the increments of maximum order of the isochromes for cracks of different lengths was found by graphical extrapolation using circular polarization in an enlarged image of the interference fringes. An optical comparator was used to determine the position of the orders and the boundary observations. Mainly monochromatic light with wavelength 546.1 μm was used.

As a result of the experimental investigation, a dependence of the stress-intensity factor $k_{\exp}$ on the crack half-length a has been constructed in dimensionless form for the same strip material (Fig. 3). It is seen from an examination of the experimental curve that $k_{\exp}$ diminishes with the diminution of the crack length, where the change is not proportional to $\sqrt{a}$, as is the result from classical theory.

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COMPRESSIVE STRENGTH OF MATERIALS

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A state of compression in material, rocks, machine parts, and structural elements is evidently very common. As a measure of this state, the concept of uniaxial compressive strength was introduced and considered as a fundamental characteristic of the material. One might also suppose that a material is fractured when the maximum tangential stress reaches the breaking point, which turns out to be half as large as the compressive strength. It should be noted that the compressive strength depends rather strongly on many factors, including the shape of the sample, its dimensions and volume, and end conditions [1, 2]. Experimenters long ago concluded that the uniaxial compressive strength is not a characteristic of the material [3, 4]. Actually, the description of the fracture patterns of samples recorded so far [2, 3, 5] with the formation of oblique fracture surfaces or surfaces of discontinuity by a coaxial compressive load can more probably be related to the shear or tensile stresses than to the compressive load. In addition, experimental and theoretical studies have shown that in tests which at first glance seem simple, the pattern of the stressed state is complex, not one-dimensional, and varies with the experimental conditions [6-9].

A number of results were obtained in [9] from a study of the stressed state of a sample under plain strain by investigating the effects of dimensions, the correlation of properties, and the role of the inserts.

In the present paper we present a numerical study of the state of stress of tubular samples and consider certain characteristic features which are interesting and important from the point of view of understanding the significance of compressive strength. The calculation was performed by the method of finite elements, using triangular-shaped elements [10]. In a cross section of a tubular sample along a meridional plane shown in Fig. 1, D and d are the outside and inside diameters, L is the length of the sample, and L_1 is the length of the inserts. The division into elements is shown in the upper symmetric half ABFE. The stressed state is characterized by the four components of the stress tensor σ_z, σ_r, σ_θ, and τ_{rz}. Since the pattern is symmetric with respect to a quarter of the cylinder (OO' is the axis of symmetry and AB is the plane of symmetry), we set up and solve the problem for the region ABFE. We formulate the following boundary conditions:

$$\begin{gathered} v = 0,\ \tau_{rz} = 0;\ z = L/2,\ d/2 < r < D/2; \\ u = 0,\ v = \text{const};\ z = L,\ d/2 < r < D/2; \\ \tau_{rz} = 0,\ \sigma_r = 0;\ r = d/2,\ D/2,\ L/2 < z < L. \end{gathered} \tag{1}$$

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